Formulation and asymptotic properties of the bifurcation ratio in Horton's law for the equiprobable binary tree model

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The bifurcation ratio for the equiprobable binary tree model is formulated. We obtain the exact expression of the kth moment of the second-order streams. We also obtain a recursive equation between rth and $(r+1)$th order streams. Horton's law is confirmed numerically by calculating this recursive equation and asymptotic properties of the bifurcation ratio are discussed.

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I. INTRODUCTION

Branching patterns are widely found throughout nature (e.g., Ref. [1] or [2]). Some of them have been characterized from the viewpoint of self-similarity or self-affinity, and the fractal dimension of these patterns has also been measured. Fractality of some branching patterns has been proved experimentally [3], theoretically [4], or numerically [5].

In order to characterize branching patterns, a method which reflects hierarchical structures of branching patterns is also needed. One such method, the ordering of river networks, was originally introduced by Horton [6] and modified by Strahler [7]. Their method, called "Horton-Strahler ordering," is defined as follows. (i) A stream originating from a source has order 1. (ii) Joining of two streams of order r arises a stream of order $r+1$. (iii) Merging of streams of order r_1 and r_2 (let $r_1 > r_2$) forms a stream of order r_1. Then, Horton's law of stream numbers [8] is stated as

$$\frac{S_r}{S_{r+1}} \approx 4, \tag{1}$$

where S_r represents the number of streams of order r and "$\approx$" denotes an approximate equality.

The Horton-Strahler analysis is valid for branching patterns containing no islands or junctions of more than three streams [8,9]; the branching structures of these patterns are topologically modeled by binary-tree graphs. In this sense, the outlet of a river network corresponds to the root of a binary tree and a source corresponds to a leaf (i.e., degree-one node except the root). In fact, many branching patterns have been analyzed by Horton-Strahler ordering (e.g., DLA patterns [10–12], bronchial trees [13,14], percolation cluster growth [15], and neural networks [16]). As the case of river networks, we also use "source" instead of "leaf" of a binary tree. The number n of sources is called "magnitude" and "stream" means the maximal connected path of incident nodes of the same order.

According to the notation of Wang and Waymire [17], let Ω_n be the sample space of topologically distinct binary trees of magnitude n. The cardinality of Ω_n is given by

$$\sharp\Omega_n = c_{n-1} \equiv \frac{1}{n}\binom{2n-2}{n-1},$$

where "$\sharp$" denotes the cardinality of a set and c_{n-1} is called the $(n-1)$th Catalan number. For each $T \in \Omega_n$, the uniform probability measure P_n assigns probability $|\Omega_n|^{-1}$. This model was originally introduced by Shreve [8], which is called the S model by Werner [18] or equiprobable binary tree by Devroye and Kruszewski [19]. Hereafter we adopt $S_{r,n}$ as a random variable on Ω_n, which indicates the number of rth order streams. By using the averaged value of $S_{r,n}$ over Ω_n [which is denoted by $E(S_{r,n})$], Horton's law of stream numbers (1) can be expressed more accurately as

$$\lim_{n\to\infty} \frac{E(S_{r,n})}{E(S_{r+1,n})} = 4. \tag{2}$$

The ratio $E(S_{r,n})/E(S_{r+1,n})$ is called "bifurcation ratio." This equiprobable model is the simplest way of selecting a binary tree of fixed magnitude.

In this paper, we calculate the kth moment of $S_{2,n}$ on the equiprobable binary trees. To do this, we introduce "hierarchical hanging" of the perfect binary trees by hanging the perfect binary trees sequentially. The result is expressed by using the hypergeometric function. Then, we derive a recursive equation between the kth moment of the rth and $(r+1)$th order streams. By using this recursive equation, the Horton's law of stream number is proved numerically and analytically.

II. MODIFIED BINARY TREE AND PERFECT BINARY TREE

For the convenience of the following calculation, let an "imaginary" node join to the root of a binary tree. We assume that this node has no Horton-Strahler number (see Fig. 1).

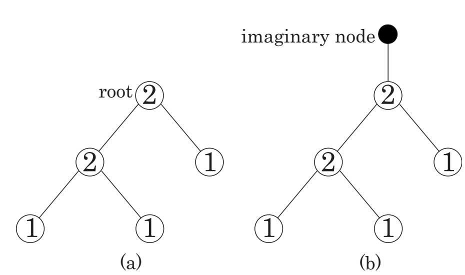

FIG. 1. A binary tree of magnitude 3 ($n=3$). (a) Ordinary binary tree. (b) Modified binary tree. The number on each node shows its own Horton-Strahler number.

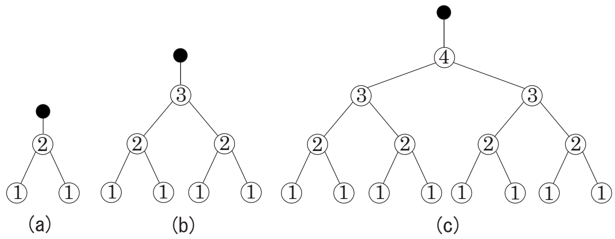

FIG. 2. Some examples of a perfect binary tree. (a) 2-perfect binary tree, (b) 3-perfect binary tree, and (c) 4-perfect binary tree.

There is a special type of a binary tree called the r-perfect (or r-complete) binary tree: every source has the same depth (=distance from the root) $r-1$ (see Fig. 2 for reference).

Perfect binary trees have some remarkable characteristics: (1) the magnitude of a r-perfect binary tree is 2^{r-1}, (2) the Horton-Strahler number of a r-perfect binary tree is r, and (3) $S(T)<r$, for any binary tree T of a magnitude less than 2^{r-1}. From these features, the r-perfect binary tree is found to be a minimal structure which contains streams of rth order.

III. CALCULATION OF THE SECOND ORDER STREAMS

As mentioned above, a random variable $S_{2,n}$ denotes the number of second-order streams in the binary tree of magnitude n. Notice that the range of $S_{2,n}$ is $\{1,2,\ldots,\lfloor n/2\rfloor\}$ for $n\geqslant 2$, where $\lfloor\cdots\rfloor$ is the floor function.

In this section, let us calculate the value

$$N(n,m)\equiv \sharp\{T\in\Omega_n|S_{2,n}(T)=m\},$$

which represents the number of binary trees in Ω_n with m second-order streams. Using this, an average and kth moment of $S_{2,n}$ are expressed as

$$E(S_{2,n})=\sum_{m=1}^{\lfloor n/2\rfloor} mP_n(S_{2,n}=m)=\frac{1}{c_{n-1}}\sum_{m=1}^{\lfloor n/2\rfloor} mN(n,m),$$

$$M_k(S_{2,n})=\sum_{m=1}^{\lfloor n/2\rfloor} m^kP_n(S_{2,n}=m)=\frac{1}{c_{n-1}}\sum_{m=1}^{\lfloor n/2\rfloor} m^kN(n,m), \quad (3)$$

where $M_k(\cdots)$ denotes the kth moment over Ω_n. A second-order stream arises from merging of two streams of order 1. Then, when a binary tree $T\in\Omega_n$ has m second-order streams, m pairs of two sources merge to produce the second-order streams and the rest of the $n-2m$ sources attach to second- or higher-order streams.

Then, a procedure to make a binary tree of magnitude n having m second-order streams is given as follows (see Fig. 3 for reference). (a) Make a binary tree of magnitude m. (b) Replace every source with a 2-perfect binary tree. (c) "Hang" $n-2m$ sources from edges of this binary tree, except for the edges of order 1. (d) Change each hanging point into a node of a binary tree.

According to the above procedure, $N(n,m)$ is given by

$$N(n,m)=2^{n-2m}c_{m-1}\binom{n-2}{n-2m}=2^{n-2m}\frac{(n-2)!}{(n-2m)!m!(m-1)!}, \quad (4)$$

where the factor c_{m-1} represents the number of topologically distinct binary trees of magnitude m, $\binom{n-2}{n-2m}$ represents the number of different ways of $n-2m$ sources selecting which edge to hang, and 2^{n-2m} represents the multiplicity of hanging from the right or the left for each hanging source. Equation (4) is essentially the same as that obtained by Shreve [8]. However, it is advantageous that our method for calculation of $N(n,m)$ can be generalized easily to the calculation of higher-order streams, as shown below.

From Eqs. (3) and (4), the kth moment of $S_{2,n}$ is expressed as

$$M_k(S_{2,n})=\frac{1}{c_{n-1}}\sum_{m=1}^{\lfloor n/2\rfloor} m^k2^{n-2m}\frac{(n-2)!}{(n-2m)!m!(m-1)!}.$$

By using the function $G(x)$ defined by

$$G(x)=\frac{1}{c_{n-1}}\sum_{m=1}^{\lfloor n/2\rfloor} 2^{n-2m}\frac{(n-2)!}{(n-2m)!m!(m-1)!}x^m,$$

the kth moment of $S_{2,n}$ is rewritten as

$$M_k(S_{2,n})=\left.\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^kG(x)\right|_{x=1}.$$

It is noted that the function $G(x)$ is calculated as

$$G(x)=\frac{2^{n-2}}{c_{n-1}}xF\left(\frac{2-n}{2},\frac{3-n}{2};2;x\right),$$

where $F(\alpha,\beta;\gamma;x)$ is the Gauss hypergeometric function defined by

$$F(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)\Gamma(\gamma+n)}\frac{x^n}{n!}.$$

Therefore, $M_k(S_{2,n})$ is obtained by

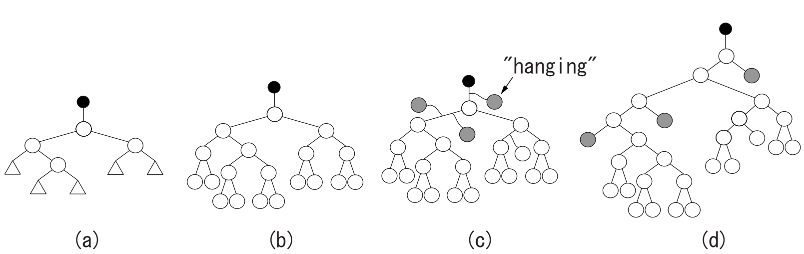

FIG. 3. Procedure to make a binary tree in the case of $n=14$, $m=5$. (a) A binary tree of magnitude 5, where $\triangle$ represents leaf. (b) Replacing $\triangle$ nodes with two perfect binary trees. (c) Hanging $4(=n-2m)$ nodes (gray nodes are hanging nodes). (d) Remaking the binary tree.

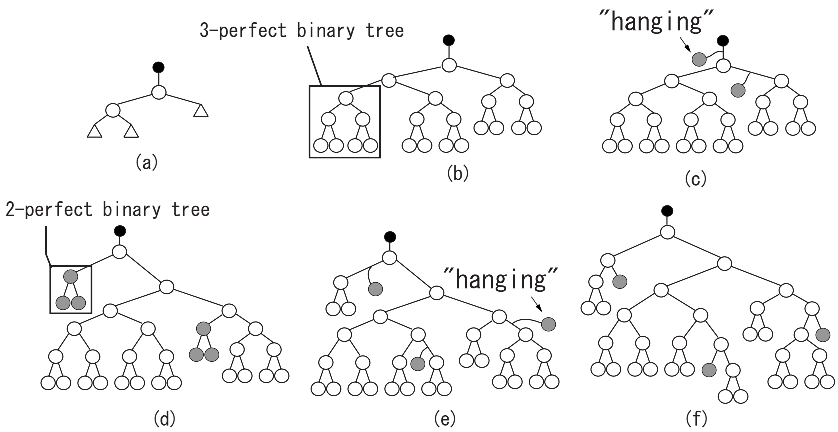

FIG. 4. Procedure to make a binary tree in the case of $n=19$, $n_2=8$, $n_3=3$. (a) A binary tree of magnitude $3(=n_3)$, where $\triangle$ represents leaf. (b) Replacing $\triangle$ nodes with 3-perfect binary trees. (c) Hanging $2(=n_2-2n_3)$ nodes. (d) Replacing hanging nodes with 2-perfect binary trees. (e) Hanging $3(=n-2n_2)$ nodes. (f) Remaking the binary tree.

$$M_k(S_{2,n}) = \frac{2^{n-2}}{c_{n-1}}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k x\ F\left(\frac{2-n}{2},\frac{3-n}{2};2;x\right)\Bigg|_{x=1} = \frac{2^{n-2}}{c_{n-1}}\left(\frac{\mathrm{d}}{\mathrm{d}x}x\right)^k F\left(\frac{2-n}{2},\frac{3-n}{2};2;x\right)\Bigg|_{x=1}. \quad (5)$$

For instance, using the Gauss formula

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad (6)$$

yields

$$F\left(\frac{2-n}{2},\frac{3-n}{2};2;1\right) = \frac{(2n-2)!}{2^{n-2}(n-1)!n!} = \frac{c_{n-1}}{2^{n-2}}. \quad (7)$$

And by using the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x), \quad (8)$$

the average and the second moment of $S_{2,n}$ are expressed as

$$E(S_{2,n}) = \frac{n(n-1)}{2(2n-3)},$$

$$M_2(S_{2,n}) = \frac{n(n-1)(n^2-n-4)}{4(2n-3)(2n-5)}. \quad (9)$$

Then, the variance of $S_{2,n}$ is given by

$$\mathrm{var}(S_{2,n}) = M_2(S_{2,n}) - E(S_{2,n})^2 = \frac{n(n-1)(n-2)(n-3)}{2(2n-3)^2(2n-5)}. \quad (10)$$

It is noted that Eqs. (9) and (10) are equivalent to the result by Werner [18] and Wang and Waymire [17].

IV. CALCULATION OF THE rth ORDER STREAMS

In this section, let us calculate the value

$$N(n,n_2,n_3,\ldots,n_r) \equiv \sharp\{T \in \Omega_n | S_{i,n}(T)=n_i, i=2,3,\ldots,r\},$$

which represents the number of binary trees in Ω_n with n_2 second order streams, n_3 third order streams,…, and n_r rth order streams.

It is noted that a minimal structure containing rth order streams is the r-perfect binary tree. Similar to the procedure described in the previous section, a procedure to make a binary tree of magnitude n having n_i ith ordered streams ($i=2,3,\ldots,r$) is as follows (see Fig. 4 for reference): (a) make a binary tree of magnitude n_r, (b) replace every source with r-perfect binary tree, (c) "hang" $n_{r-1}-2n_r$ sources from the edge of this binary tree, except for the edges of order less than r, (d) replace every hanging source with $(r-1)$-perfect binary tree, (e) hang $n_{r-2}-2n_{r-1}$ sources from the edge of the binary tree, except for the edges of order less than $r-1$, then replace every hanging source with $(r-2)$-perfect binary tree, and so on. In short, perfect binary trees are hung hierarchically through this method.

According to the above procedure, $N(n,n_2,n_3,\ldots,n_r)$ is calculated as

$$N(n,n_2,n_3,\ldots,n_r) = c_{n_r-1}\prod_{i=1}^{r-1} 2^{n_i-2n_{i+1}}\binom{n_i-2}{n_i-2n_{i+1}},$$

where $n_1 \equiv n$. The factor c_{n_r-1} represents the number of topologically distinct binary trees of magnitude n_r, $\binom{n-2}{n_i-2n_{i+1}}$ represents the number of different ways of n_i-2n_{i+1} sources selecting which edge to hang, and $2^{n_i-2n_{i+1}}$ represents the multiplicity of hanging from the right or the left for each hanging source ($i=r-1,r-2,\ldots,2$). The case where $n_r=1$ is consistent with the result by Shreve [8].

Notice that the ith order stream is merely formed by joining two streams of $(i-1)$th order. If the values n and n_r are fixed, $2^{r-i}n_r \leq n_i \leq \lfloor n_{i-1}/2 \rfloor$ holds for $i=2,3,\ldots,r$. Hence

$$\sharp\{T \in \Omega_n | S_{r,n}(T)=n_r\} = \sum_{n_2=2^{r-2}n_r}^{\lfloor n/2 \rfloor}\ \sum_{n_3=2^{r-3}n_r}^{\lfloor n_2/2 \rfloor} \cdots \sum_{n_{r-1}=2n_r}^{\lfloor n_{r-2}/2 \rfloor} N(n,n_2,n_3,\ldots,n_r)$$

is obtained. Since $0 \leq n_r \leq \lfloor n/2^{r-1} \rfloor$ is also confirmed, the kth moment of $S_{r,n}$ is expressed as

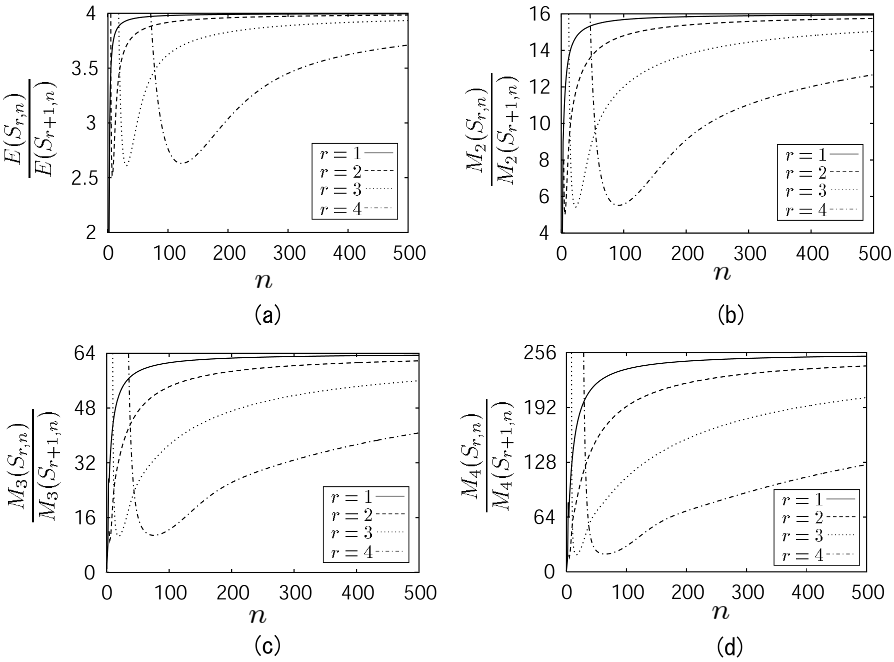

FIG. 5. kth moment ratio of $S_{r,n}$ to $S_{r+1,n}$ for $k=1$ (a), $k=2$ (b), $k=3$ (c), and $k=4$ (d).

$$M_k(S_{r,n}) = \frac{1}{c_{n-1}}\sum_{n_r=1}^{\lfloor n/2^{r-1}\rfloor} n_r^k \sharp\{T\in\Omega_n | S_{r,n}(T)=n_r\}$$
$$= \frac{1}{c_{n-1}}\sum_{n_r=1}^{\lfloor n/2^{r-1}\rfloor}\sum_{n_2=2^{r-2}n_r}^{\lfloor n/2\rfloor}\sum_{n_3=2^{r-3}n_r}^{\lfloor n_2/2\rfloor}\cdots\sum_{n_{r-1}=2n_r}^{\lfloor n_{r-2}/2\rfloor} n_r^k c_{n_r-1}$$
$$\times\prod_{i=1}^{r-1}2^{n_i-2n_{i+1}}\binom{n_i-2}{n_i-2n_{i+1}}. \quad (11)$$

After some calculation (see Appendix A), we have

$$M_k(S_{r,n}) = \frac{1}{c_{n-1}}\sum_{m=2^{r-2}}^{\lfloor n/2\rfloor}2^{n-2m}\binom{n-2}{n-2m}c_{m-1}M_k(S_{r-1,m})$$
$$= \frac{1}{c_{n-1}}\sum_{m=2^{r-2}}^{\lfloor n/2\rfloor}\frac{2^{n-2m}(n-2)!}{(n-2m)!m!(m-1)!}M_k(S_{r-1,m}). \quad (12)$$

Regarding Eq. (12) as a recursive equation about r, $M_k(S_{2,n}), M_k(S_{3,n}),\ldots$, can be calculated iteratively from the initial condition $M_k(S_{1,n})=n^k$. The results of calculation of this equation are shown as the ratio $M_k(S_{r,n})/M_k(S_{r+1,n})$ in Fig. 5 for $k=1,2,3$, and 4. From this figure, it is suggested that

$$\lim_{n\to\infty}\frac{M_k(S_{r,n})}{M_k(S_{r+1,n})} = 4^k, \quad (13)$$

and that the speed of convergence of $M_k(S_{r,n})/M_k(S_{r+1,n})$ is more slowly for larger r. These properties are demonstrated analytically in the following section.

V. ASYMPTOTIC PROPERTIES

In this section, we discuss some asymptotic properties of $M_k(S_{r,n})/M_k(S_{r+1,n})$ for large n. Using the formulas (8) repeatedly, we obtain

$$F^{(k)}(\alpha,\beta;\gamma;x) = \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+k)}\times F(\alpha+k,\beta+k;\gamma+k;x).$$

Then, using the Gauss formulas (6) yields

$$F^{(k)}(\alpha,\beta;\gamma;1)$$
$$= \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+k)}\frac{\Gamma(\gamma+k)\Gamma(\gamma-\alpha-\beta-k)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$
$$= \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma-\alpha-\beta-k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$
$$= \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma-\alpha-\beta-k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}F(\alpha,\beta;\gamma;1), \quad (14)$$

where $F^{(k)}(\alpha,\beta;\gamma;1)\equiv F^{(k)}(\alpha,\beta;\gamma;x)|_{x=1}$ hereafter.

Specifically, in the case of $\alpha=(2-n)/2$, $\beta=(3-n)/2$ and $\gamma=2$, Eq. (14) becomes

$$F^{(k)}\left(\frac{2-n}{2},\frac{3-n}{2};2;1\right)$$
$$= \frac{1}{2^k}\frac{(n-2)!(2n-2k-3)!!}{(n-2k-2)!(2n-3)!!}F\left(\frac{2-n}{2},\frac{3-n}{2};2;1\right). \quad (15)$$

From Eq. (5), the kth moment of $S_{2,n}$ has the form

$$M_k(S_{2,n}) = \frac{2^{n-2}}{c_{n-1}} \sum_{l=0}^{k} a_{l,k} F^{(l)}\left(\frac{2-n}{2}, \frac{3-n}{2}; 2; 1\right),$$

where $a_{\ell,k}$ is a constant, and $a_{k,k}=1$ and $a_{k,k-1}=k(k+1)/2$ is proved inductively. Furthermore, by using Eqs. (7) and (15),

$$M_k(S_{2,n}) = \sum_{l=0}^{k} a_{l,k} \frac{1}{2^l} \frac{(n-2)!(2n-2l-3)!!}{(n-2l-2)!(2n-3)!!} \qquad (16)$$

is derived.

Then, the asymptotic form of $M_k(S_{2,n})$ is derived as

$$M_k(S_{2,n}) = \frac{n^k}{4^k}\left(1 + \frac{k^2}{2n} + O(n^{-2})\right). \qquad (17)$$

Calculation from Eq. (16) to Eq. (17) is explained in Appendix B.

Similarly, in the case of $r=3$, $M_k(S_{3,n})$ has the form

$$\begin{aligned} M_k(S_{3,n}) &= \frac{1}{c_{n-1}} \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{2^{n-2m}(n-2)!}{(n-2m)!m!(m-1)!} M_k(S_{2,m}) \\ &= \frac{1}{c_{n-1}} \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{2^{n-2m}(n-2)!}{(n-2m)!m!(m-1)!} \frac{m^k}{4^k} \\ &\quad \times \left(1 + \frac{k^2}{2m} + O(m^{-2})\right) \\ &= \frac{1}{4^k} M_k(S_{2,n}) + \frac{k^2}{2} M_{k-1}(S_{2,n}) + O(n^{k-2}) \\ &= \left(\frac{n}{4^2}\right)^k \left(1 + \frac{3k^2}{2n}\right) + O(n^{k-2}), \end{aligned} \qquad (18)$$

where we note that

$$\frac{1}{c_{n-1}} \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{2^{n-2m}(n-2)!}{(n-2m)!m!(m-1)!} m^l = M_l(S_{2,n}).$$

Furthermore, from Eqs. (17) and (18),

$$M_k(S_{r,n}) = \left(\frac{n}{4^{r-1}}\right)^k \left(1 + \frac{\mu_r k^2}{2n}\right) + O(n^{k-2}) \qquad (19)$$

is suggested. Here μ_r is a constant determined as follows. From Eq. (17), we have $\mu_2=1$ immediately. Substituting Eq. (19) into Eq. (12), we obtain

$$\begin{aligned} M_k(S_{r+1,n}) &= \frac{1}{c_{n-1}} \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{2^{n-2m}(n-2)!}{(n-2m)!m!(m-1)!} \left(\frac{m}{4^{r-1}}\right)^k \\ &\quad \times \left(1 + \frac{\mu_r k^2}{2m} + O(m^{-2})\right) \\ &= \left(\frac{n}{4^r}\right)\left(1 + \frac{(4\mu_r + 1)k^2}{2n}\right) + O(n^{k-2}). \end{aligned}$$

On the other hand,

$$M_k(S_{r+1,n}) = \left(\frac{n}{4^r}\right)^k \left(1 + \frac{\mu_{r+1} k^2}{2n}\right) + O(n^{k-2})$$

is obtained from the definition of μ_{r+1}. Thus we have

$$\mu_2 = 1,$$

$$\mu_{r+1} = 4\mu_r + 1,$$

and solve it easily as

$$\mu_r = \frac{1}{3}(4^{r-1} - 1).$$

Consequently, the asymptotic form of $M_k(S_{r,n})$ is expressed as

$$M_k(S_{r,n}) = \left(\frac{n}{4^{r-1}}\right)^k \left(1 + \frac{(4^{r-1}-1)k^2}{6n}\right) + O(n^{k-2}).$$

We also derive

$$\begin{aligned} \frac{M_k(S_{r,n})}{M_k(S_{r+1,n})} &= \frac{\left(\frac{n}{4^{r-1}}\right)^k \left[1 + \frac{(4^{r-1}-1)k^2}{6n}\right]}{\left(\frac{n}{4^r}\right)^k \left[1 + \frac{(4^{r-1}-1)k^2}{6n}\right]} + O(n^{-2}) \\ &= 4^k \left(1 - \frac{4^{r-1}k^2}{2n}\right) + O(n^{-2}). \end{aligned} \qquad (20)$$

Therefore Eq. (13) is proved. This equation is an extended form of Horton's law. Moreover, the speed of convergence of $M_k(S_{r,n})/M_k(S_{r+1,n})$ is more slow for larger r, because of the term $4^{r-1}k^2$ in Eq. (20).

Especially in the case $k=1$, Eq. (20) turns into

$$\frac{E(S_{r,n})}{E(S_{r+1,n})} = 4 - \frac{4^r}{2n} + O(n^{-2}). \qquad (21)$$

Then, the Horton's law of stream number

$$\lim_{n\to\infty} \frac{E(S_{r,n})}{E(S_{r+1,n})} = 4 \qquad (22)$$

is reproduced.

From Eq. (21), we define $R(n)$ as

$$R(n) \equiv \frac{1}{4^{r-1}}\left(4 - \frac{E(S_{r,n})}{E(S_{r+1,n})}\right) = \frac{2}{n} + O(n^{-2}). \qquad (23)$$

It is noted that $R(n)$ is independent of r if $O(n^{-2})$ terms are neglected. $R(n)$ obtained from the data in Fig. 5(a) is illustrated in Fig. 6.

VI. DISCUSSION AND SUMMARY

In the present paper, we calculated $M_k(S_{2,n})$ by hanging the perfect binary trees sequentially [Eq. (5)] and derived the recursive equation (12). Then, we calculated this equation numerically (Fig. 5). Moreover, we proved the Horton's law of stream numbers (22) and (21) and extended relations (13) and (20) by using this recursive equation. The equiprobable model is a special class of binary-tree selection. So our result is not applicable directly to every class of branching pattern since our result is all based on the equiprobable binary trees [20].

It is noted that the exact expression of $M_k(S_{2,n})$ [Eq. (5)] is an extension of the result by Werner [18]. And Eq. (13) is

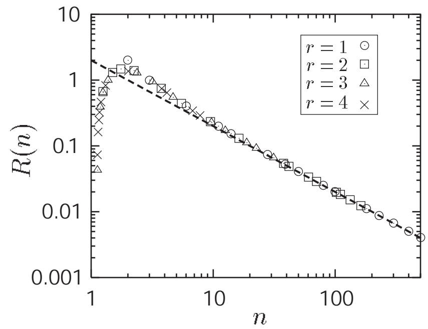

FIG. 6. Log-log plot of $R(n)$ for $r=1$, 2, 3, and 4. Data points are constructed from Fig. 5(a), and thinned out randomly. The dashed line indicates $R(n)=\frac{2}{n}$.

natural generalization of the Horton's law (2). Viennot [21] mentioned that Moon [22] implied Eq. (21) using generating function method. And related results have also been obtained by generating function method [17]. Although our method is not based on generating function, our result partially contains those obtained by generating function method. There are several results about the analysis of the higher-order (or rth order) streams [22–25]. Our method can be extended naturally to the case of higher-order moments and higher-order streams. The above procedure of successive hanging of perfect binary trees is viewed as an addition of fine structure successively to coarse structure. This idea is similar to the concept of renormalization [26,27]. We expect that our method can be reconsidered from the viewpoint of renormalization.

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APPENDIX A: DERIVATION OF EQ. (12) FROM EQ. (11)

Let us derive Eq. (12) from Eq. (11). By changing the order of summation

$$\sum_{n_r=1}^{\lfloor n/2^{r-1}\rfloor} \sum_{n_2=2^{r-2}n_r}^{\lfloor n/2\rfloor} = \sum_{n_2=2^{r-2}}^{\lfloor n/2\rfloor} \sum_{n_r=1}^{\lfloor n/2^{r-2}\rfloor},$$

$$\sum_{n_r=1}^{\lfloor n/2^{r-2}\rfloor} \sum_{n_3=2^{r-3}n_r}^{\lfloor n_2/2\rfloor} = \sum_{n_3=2^{r-3}}^{\lfloor n_2/2\rfloor} \sum_{n_r=1}^{\lfloor n/2^{r-3}\rfloor}, \ldots,$$

we can rewrite Eq. (11) as

$$\begin{aligned} M_k(S_{r,n}) &= \frac{1}{c_{n-1}} \sum_{n_2=2^{r-2}}^{\lfloor n/2\rfloor} \sum_{n_3=2^{r-3}}^{\lfloor n_2/2\rfloor} \cdots \sum_{n_r=1}^{\lfloor n_{r-1}/2\rfloor} n_r^k c_{n_r-1} \\ &\quad \times \prod_{i=1}^{r-1} 2^{n_i-2n_{i+1}} \binom{n_i-2}{n_i-2n_{i+1}} \\ &= \frac{1}{c_{n-1}} \sum_{n_2=2^{r-2}}^{\lfloor n/2\rfloor} 2^{n-2n_2} \binom{n-2}{n-2n_2} \\ &\quad \times \sum_{n_3=2^{r-3}}^{\lfloor n_2/2\rfloor} 2^{n_2-2n_3} \binom{n_2-2}{n_2-2n_3} \cdots \\ &\quad \times \sum_{n_r=1}^{\lfloor n_{r-1}/2\rfloor} n_r^k c_{n_r-1} 2^{n_{r-1}-2n_r} \binom{n_{r-1}-2}{n_{r-1}-2n_r}. \end{aligned} \tag{A1}$$

Similarly, $M_k(S_{r-1,n_2})$ is written as

$$\begin{aligned} M_k(S_{r-1,n_2}) &= \frac{1}{c_{n_2-1}} \sum_{n_3=2^{r-3}}^{\lfloor n_2/2\rfloor} 2^{n_2-2n_3} \binom{n_2-2}{n_2-2n_3} \cdots \\ &\quad \times \sum_{n_r=1}^{\lfloor n_{r-1}/2\rfloor} n_r^k c_{n_r-1} 2^{n_{r-1}-2n_r} \binom{n_{r-1}-2}{n_{r-1}-2n_r}. \end{aligned} \tag{A2}$$

Comparing Eqs. (A1) and (A2), we have Eq. (12) [note that the factor $\Sigma_{n_3=2^{r-3}}^{\lfloor n_2/2\rfloor}\cdots$ commonly appears in Eqs. (A1) and (A2)].

APPENDIX B: DERIVATION OF EQ. (17) FROM EQ. (16)

Considering the rough approximation

$$\begin{aligned} &\frac{1}{2^l}\frac{(n-2)!(2n-2l-3)!!}{(n-2l-2)!(2n-3)!!} \\ &\quad = \frac{1}{2^l}\frac{\overbrace{(n-2)(n-3)\cdots(n-2l-1)}^{2l}}{\underbrace{(2n-3)(2n-5)\cdots(2n-2l-1)}_{l}} \simeq \frac{n^l}{4^l}, \end{aligned}$$

dominant terms in Eq. (16) is shown as

$$\begin{aligned} M_k(S_{2,n}) &= a_{k,k}\frac{1}{2^k}\frac{(n-2)!(2n-2k-3)!!}{(n-2k-2)!(2n-3)!!} \\ &\quad + a_{k-1,k}\frac{1}{2^{k-1}}\frac{(n-2)!(2n-2k-1)!!}{(n-2k)!(2n-3)!!} + O(n^{k-2}) \\ &= \frac{1}{2^k}\frac{(n-2)!(2n-2k-3)!!}{(n-2k-2)!(2n-3)!!} \\ &\quad + \frac{k(k+1)}{2^k}\frac{(n-2)!(2n-2k-1)!!}{(n-2k)!(2n-3)!!} + O(n^{k-2}). \end{aligned} \tag{B1}$$

By using the expansion

$$\frac{1}{2n-l} = \frac{1}{2n}\left(1 + \frac{l}{2n} + O(n^{-2})\right),$$

we have

$$\frac{(n-2)!(2n-2k-3)!!}{(n-2k-2)!(2n-3)!!} = \frac{n^k}{2^k}\left\{1 - \left(\frac{3}{2}k^2 + 2k\right)\frac{1}{n}\right\} + O(n^{k-2}), \tag{B2}$$

$$\frac{(n-2)!(2n-2k-1)!!}{(n-2k)!(2n-3)!!} = \frac{n^{k-1}}{2^{k-1}} + O(n^{k-2}). \tag{B3}$$

By substituting Eqs. (B2) and (B3) into Eq. (B1), Eq. (17) is derived.

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